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Automorphism groups of pure integral octonions

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Abstract. Matrix representation of the automorphism group of pure integral octonions constituting the root system of E_7 is constructed. It is shown that it is a finite subgroup of the exceptional group of G_2 of order 12096, called the adjoint Chevalley group $G_2(2)$. Its four maximal subgroups of orders 432, 192, 192′ and 336 preserve, respectively, the octonionic root systems of E_6, $SO(12)$, $SU(2)^3 \times SO(8)$ and $SU(8)$. It is also shown explicitly that the full automorphism group of the pure octonions $\pm e_i$ ($i = 1, \ldots, 7$) constituting the roots of $SU(2)^7$ is a group of order 1344. Possible implications in physics are discussed.

1. Introduction

It is a well known phenomena that the finite subgroups of the rotation group $SO(3)$ play prominent roles in the molecular structures of atoms. Crystallography has important applications of the discrete subgroups of the rotation group adjoined by the Euclidean translations. Ordinary Lie groups or super Lie groups had great impact both in nuclear and particle physics as a classification symmetry and/or gauge symmetry.

Simple finite groups have, for nearly a century, been a central resaerch area in mathematics and their classification has recently been completed [1]. They are the Lie-type groups, alternating groups and the 26 sporadic groups. There is no evidence as to their role in physics. However, it has been already noted that the finite subgroups of $SU(2)$ [2], which are the double covers of the finite subgroups of the ordinary rotation groups, are related to the A–D–E classification of the Wess–Zumino–Witten models based on the corresponding affine Lie algebras [3]. The $SL_2(7)$ group of order 336, which is the double cover of the finite subgroup of $SU(3)$ of order 168, has some relation with rational conformal field theory based on the G_2 affine Lie algebra [4, 5]. A detailed study regarding the group $PSL_2(7)$ of order 168, which is isomorphic to the projective modular group on the finite field with seven elements, has been extensively worked out by Bauer and Itzykson [5].

The automorphism group of the octonion algebra is the exceptional Lie group G_2 [6], and its finite subgroups have been studied [7]. In a recent paper, it has been shown that a finite subgroup $G_2(2)$ (adjoint Chevalley group) of the exceptional Lie group G_2 can be realized as the automorphism group of the octonionic root system of E_7 [8]. Being a subalgebra of E_8 possessing an octonionic root system which is closed under octonion multiplication, the E_7 root system has a presentation in terms of pure integral octonions [9].

In this paper we work out the seven-dimensional irreducible representation of $G_2(2)$, of order 12096, and find the Lie subalgebra structures of E_7 corresponding to the maximal subgroups of $G_2(2)$. This work is an extension of an earlier paper by Karsch and Koca [8] and makes the topic more accessible for physicists. We organize the paper as follows. In section 2 we briefly discuss the construction of the E_8 root system with octonions and give explicit roots of E_7 in terms of pure octonions. We describe a method to obtain the seven-dimensional irreducible representation of $G_2(2)$ and specify three generators of the group. In section 3 we obtain the maximal subgroups of $G_2(2)$ of orders 436, 192, 192′ and 336 preserving, respectively, the octonionic root system of Lie algebras E_6, $SO(12)$, $SU(2)^3\times SO(8)$, and $SU(8)$. In section 4 we discuss the full automorphism group of order 1344 of pure octonions of $\pm e_i$ $(i=1,\ldots,7)$ and study its maximal subgroups [11] (see also H S M Coxeter in [9]). Finally, in section 5 we discuss our results and make remarks concerning the relations of $G_2(2)$ and the group 1344 with the other simple finite groups.

2. E_7 root system with pure integral octonions

In earlier publications [10] it was shown that the root system of F_4 can be described with quaternionic sets $A_\alpha(\alpha=0,1,2,3)$ where a_α are given by

$$\begin{array}{cccc} A_0 & A_1 & A_2 & A_3 \\ \hline \pm1,\pm e_1,\pm e_2,\pm e_3 & \frac{1}{2}(\pm1\pm e_1) & \frac{1}{2}(\pm1\pm e_2) & \frac{1}{2}(\pm1\pm e_3) \\ \frac{1}{2}(\pm1\pm e_1\pm e_2\pm e_3) & \frac{1}{2}(\pm e_2\pm e_3) & \frac{1}{2}(\pm e_3\pm e_1) & \frac{1}{2}(\pm e_1\pm e_2) \end{array}. \tag{1}$$

Here we have $e_ie_j=-\delta_{ij}+\varepsilon_{ijk}e_k$ $(i,j,k=1,2,3)$.

Following the Cayley–Dixon procedure for the construction of octonions from quaternions

$$[A_a, A_b]=A_a+e_7A_b \qquad a,b=0,1,2,3 \tag{2}$$

where $e_7^2=-1$ and $e_4=e_7e_1$, $e_5=e_7e_2$, $e_6=e_7e_3$, we obtain the octonionic roots of E_8 as follows:

$$E_8\colon [A_0,0],\ [0,A_0],\ [A_1,A_1],\ [A_2,A_3],\ [A_3,A_2]. \tag{3}$$

This leads to the following decomposition of the roots under $SU(2)\times E_7$ subalgebra:

$$\begin{array}{ccc} SU(2) & E_7 & E_8/E_7\times SU(2) \\ \hline \pm1 & \pm e_i,\ \frac{1}{2}(\pm e_j\pm e_k\pm e_l\pm e_m) & \frac{1}{2}(\pm1\pm e_n\pm e_p\pm e_q) \end{array} \tag{4}$$

where the indices take the values

$$\begin{aligned} &i=1,\ldots,7\\ &jklm\colon\ 1246, 1257, 1345, 1367, 2356, 2347, 4567\\ &npq\colon\ 123, 147, 165, 245, 267, 346, 357. \end{aligned} \tag{5}$$

The set of 240 integral octonions in (4) constitutes the units of integral octonions describing the E_8 lattice. The 240 units form the non-zero roots of E_8 provided multiplied by $\sqrt{2}$ and are closed under octonion multiplications.

In our construction e_1 plays a special role. In fact, octonionic roots of E_8 similar to (3) can be obtained seven different ways, in each of which one of the octonionic units

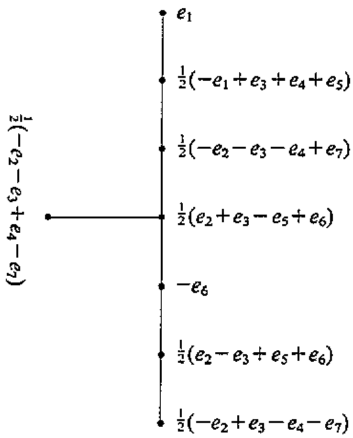

Figure 1. Extended Dynkin diagram of E_7 with pure octonionic units.

e_i ($i=1, \ldots, 7$) plays a crucial role [10]. The roots of E_7 in (4) can also be obtained from the simple roots of E_7, and the extended Dynkin diagram is given in figure 1. The simple roots of E_7 in figure 1 can be obtained by the transformation from those which are given in [8]:

$$
\begin{aligned}
& e_1 \rightarrow e_5 \\
& e_2 \rightarrow \tfrac{1}{2}(-e_2-e_3-e_4+e_7) \\
& e_3 \rightarrow \tfrac{1}{2}(e_2-e_3+e_4+e_7) \\
& e_4 \rightarrow \tfrac{1}{2}(-e_2-e_3+e_4-e_7) \\
& e_5 \rightarrow e_1 \\
& e_6 \rightarrow -e_6 \\
& e_7 \rightarrow \tfrac{1}{2}(-e_2+e_3+e_4+e_7).
\end{aligned}
$$

We make this choice of simple roots in figure 1 to represent the matrices of the group 192 with the matrix elements $\pm 1, 0$. We will also make use of this representation in the construction of the group 1344.

In order to construct the automorphism group of the pure octonions in (4) let us choose three roots

$$
\begin{aligned}
& p_1=\tfrac{1}{2}(1+e_1+e_2+e_3) \\
& p_2=\tfrac{1}{2}(1-e_1+e_2+e_3) \\
& p_3=\tfrac{1}{2}(1+e_1+e_5+e_6)
\end{aligned}
\tag{6}
$$

from the coset space $E_8/E_7 \times SU(2)$ and define three 7×7 matrices corresponding to the transformations

$$
\begin{aligned}
& e_i'=(\pm p_\alpha)e_i(\pm \bar{p}_\alpha) \equiv \sum_{j=1}^{7} (P_\alpha)_{ij} e_j \\
& \alpha=1, 2, 3.
\end{aligned}
\tag{7}
$$

Table 1. Character values of the seven-dimensional irreducible representation of $U_3(3) = G_2'(2)$.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	56	−2	3
3	63	−1	2
4	63	3	4
5	63	3	4
6	378	−1	4
7	504	0	12
8	504	0	12
9	504	2	6
10	672	1	3
11	756	−1	8
12	756	−1	8
13	864	0	7
14	864	0	7

The matrices $P_\alpha (\alpha = 1, 2, 3)$ generate a group of order 6048 which is the simple group $U_3(3) = G_2'(2)$, called the derived Chevalley group [12]. It has 14 conjugacy classes. The characters of its seven-dimensional irreducible representation are given in table 1. The matrices P_1, P_2, P_3 belong to the same conjugacy classes of order 3 with Tr $P_i = -2$. In fact, there are 56 matrices of type (7) all belonging to the same class in $G_2'(2)$. To minimize the number of matrices generating $G_2'(2)$, $G_2(2)$ and their maximal subgroups, we give seven matrices in the appendix such that various combinations of them generate the groups in question. For instance, it is possible to generate $G_2'(2)$ with the matrices B and C given in the appendix.

We note that the seven-dimensional irreducible representation of $G_2'(2)$ involves three diagonal matrices other than the unit matrix and are given by

$$\begin{aligned} &M_1 = (1, 1, 1, -1, -1, -1, -1) \\ &M_2 = (1, -1, -1, 1, -1, -1, 1) \\ &M_3 = M_1 M_2 = M_2 M_1 = (1, -1, -1, -1, 1, 1, -1) \text{ and cyclic permutations.} \end{aligned} \tag{8}$$

An inspection proved that the diagonal matrix $M_4 = (-1, 1, -1, 1, -1, 1, -1)$, $M_4^2 = I$ also preserves the octonionic roots of E_7 in (4). The matrices M_1, M_2, M_4 generate an elementary Abelian group of order 8 where we define

$$M_7 = M_4 M_1 \qquad M_6 = M_4 M_2 \qquad M_5 = M_4 M_3$$

and

$$M_i M_j = M_j M_i = M_k \qquad (ijk = 123, 147, 165, 246, 257, 354, 367). \tag{9}$$

This group plays a crucial role in the analysis of the maximal subgroups of $G_2(2)$ and the group 1344.

One can readily show that the generators of $G_2'(2)$ and M_4 satisfy the relations

$$\begin{aligned} &M_4 B M_4^{-1} = B' \\ &M_4 C M_4^{-1} = C' \end{aligned} \tag{10}$$

where $B, C, B', C' \in G_2'(2)$.

This proves that M_4 constitutes the outer automorphism of $G_2'(2)$, and $G_2(2)$ is obtained by adjoining M_4 to the generators B, C of $G_2'(2)$. Thus, we obtain the seven-dimensional irreducible representation of the adjoint Chevalley group $G_2(2)$ of order $6048 \times 2 = 12096$. We note that the adjoint Chevalley group G_2 is one of the maximal subgroups of the Weyl group of E_7 with index 240. Indeed, the Weyl group of E_7 is the direct product of the Chevalley group $SO_7(2)$ with the inversion group of order 2. Therefore, $G_2(2)$ is maximal in $SO_7(2)$ with the index 120. The matrices of $SO_7(2)$ and $G_2(2)$ are the orthogonal matrices of determinant +1[1, 13]. Character values of the seven-dimensional irreducible representations of $G_2(2)$ are shown in table 2.

Table 2. Character values of the seven-dimensional irreducible representation of $G_2(2)$.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	56	−2	3
3	63	−1	2
4	252	−1	2
5	252	3	4
6	126	3	4
7	378	−1	4
8	504	2	6
9	672	1	3
10	1008	0	12
11	1008	0	12
12	1008	0	12
13	1512	1	8
14	1512	−1	8
15	1728	0	7
16	2016	−1	6

3. Maximal subgroups of $G_2(2)$ and the related subalgebras of E_7

The simple group $G_2'(2)$ has four maximal subgroups of orders 216 (3_+^{1+2}: 8), 96($2^2.S_4$), 96′($4^2.S_3$) and 168($PSL_2(7)$) [1]. The notation in parentheses is the names of the groups and will be clarified in what follows. Double covers of four respective groups of orders 432, 192, 192′ and 336 preserve the root systems of E_6, $SO(12)$, $SU(2)^3 \times SO(8)$ and $SU(8)$.

We discuss each case separately below.

3.1. The group 3_+^{1+2}:8:2 of order 432 and the octonionic roots of E_6

$E_6 \times U(1)$ is one of the maximal Lie algebras of E_7. $U(1)$, being in the Cartan subalgebra, is represented by the zero root. 126 non-zero roots of E_7 decompose as

$126 = 72 + 27 + 27^*$, where the 72 non-zero roots of E_6 are given by

$$\begin{array}{ll} \pm e_2,\ \pm e_3,\ \pm e_5,\ \pm e_6 & \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6) \\ \pm\tfrac{1}{2}(e_1 \pm e_2 - e_4 \pm e_6) & \pm\tfrac{1}{2}(e_1 \pm e_3 - e_4 \pm e_5) \\ \pm\tfrac{1}{2}(e_1 \pm e_2 \pm e_5 - e_7) & \pm\tfrac{1}{2}(e_1 \pm e_3 \pm e_6 - e_7) \\ \pm\tfrac{1}{2}(\pm e_2 \pm e_3 + e_4 - e_7) & \pm\tfrac{1}{2}(e_4 \pm e_5 \pm e_6 - e_7). \end{array} \tag{11}$$

The 24 roots in the top line of (11) correspond to the roots of the $SO(8)$ subalgebra of E_6. The matrices of $G_2'(2)$ which leave the set of roots in (11) form a group of order 216. This group which has 13 conjugacy classes can be generated by the matrices B and C given in the appendix. The notation $3_+^{1+2}:8$ used for the group 216 has the following meaning. It is the semi-direct product of two groups: one is a cyclic group of order 8 and the other is an extraspecial group of order 27. The structures of these subgroups will be discussed elsewhere [14]. The E_6 root sytsem is also left invariant by the matrix L_4, which is not an element of $G_2'(2)$. The order of L_4 is 2 since $L_4^2 = I$. By taking the semi-direct product of $3_+^{1+2}:8$ with the cyclic group of order 2 we obtain the group $3_+^{1+2}:8:2$ of order 432. We have also checked that 3_+^{1+2} is a normal subgroup of $3_+^{1+2}:8:2$ of index 16.

The Weyl group of E_6 is the automorphic extension of $SO_6(2) \cong U_4(2) \cong {}^2A_3(2)$ [1, 13] with the cyclic group of order 2. Here $SO_6(2)$, for example, is generated by 6×6 matrices over the field with two elements preserving a non-singular quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6$.

It is interesting to note that $|W(E_6)|/|3_+^{1+2}:8:2| = 120$. We have analysed the maximal subgroups of 432 and obtained three different maximal subgroups, each of order 216. In addition to the usual maximal subgroup of $G_2'(2)$ having 13 conjugacy classes, we obtained the other two groups with 13 and 16 conjugacy classes. The second group of order 216, although possessing 13 conjugacy classes, is different from the one in $G_2'(2)$.

3.2. The group $2^2.S_4:2$ of order 192 and the root system of $SU(2) \times SO(12)$

The Lie algebra $SU(2) \times SO(12)$ is maximal in E_7. In fact the isotropy group of the highest root of E_7 is the Weyl group $SO(12)$. E_7 roots decompose under $SU(2) \times SO(12)$ roots as follows:

$$\begin{array}{ll} SU(2): & \pm e_1 \\ SO(12): & \pm e_2,\ \pm e_3,\ \pm e_4,\ \pm e_5,\ \pm e_6,\ \pm e_7 \\ & \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6) \\ & \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7) \\ & \tfrac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7). \end{array} \tag{12}$$

The remaining roots of E_7 transform as the direct product of spinor representations $(2, 32)$ of $SU(2) \times SO(12)$. Here we are concerned with the transformations which stabilize $\pm e_1$. The matrices of $G_2'(2)$ preserving (12) form a group of order 96 with 10 conjugacy classes whose generators can be taken as the matrices A, C whose elements consist of only $\pm 1, 0$. The other maximal subgroups of $G_2(2)$ generated by the other matrices also include elements $\pm\frac{1}{2}$ in addition to $\pm 1, 0$. We have checked that the group of order 96 possesses the permutation group S_4 as a subgroup and contains an elementary Abelian group generated by M_1 and M_2 as the invariant subgroup of order $2^2=4$. The notation $2^2.S_4$ implies this structure. Being a subgroup of $G_2'(2)$, the $2^2.S_4$ consisting of all matrices with elements $\pm 1, 0$ involves the diagonal matrices, M_i ($i=1,2,3$). We note that M_4 also preserves the root system in (11) so that the group can be extended to $2^2.S_4$:2 of order 192 possessing 14 conjugacy classes of all matrices with ± 1, 0 in $G_2(2)$. These 192 matrices are of the form $7=1+6$ so that the 6×6 blocks act on the octonionic units e_i ($i=2,\ldots,6$) and can be further reduced to two sets of 3×3 matrix blocks. This is then a discrete subgroup of $SU(3)$ [7]. The matrices generating S_4 can further be reduced to $7=1+3+3'$ so that two blocks of 3×3 matrices acting on the sets (e_2, e_4, e_5) and (e_3, e_6, e_7) which correspond to two different three-dimensional irreducible representations of S_4. In fact, the whole group involving S_4 (finite subgroup of $SO(3)$) can be reduced to $7=1+3+3'$. Interestingly enough, $2^2.S_4$:2 is isomorphic to the Weyl group of $SO(8)$. The $SO(12)$ roots in (11) can be regarded as the union of three $SO(8)_a$ ($a=1,2,3$) roots. One of the S_3 subgroups of $2^2.S_4$:2 permutes these root systems amongst themselves. We have checked that the group of order 192 has three maximal subgroups of orders 96. In addition to the usual one which is involved in $G_2'(2)$ there are two more groups with 10 conjugacy classes. However, they are not isomorphic to each other.

3.3. *The group $4^2{:}S_3{:}2=4^2{:}D_6$ (192′) of order 192 and the root system $SU(2)^3\times SO(8)$*

A computer program gave two maximal groups of order 96 in $G_2'(2)$. The one with elements $\pm 1, 0$ which we have already discussed was obtained while searching for the transformations preserving the root system of $SO(12)$. The second one was given by computer calculation. This group has 16 conjugacy classes, and the matrices can be reduced to block diagonal forms of 3×3 and 4×4 matrices where the upper 3×3 matrices possess elements $\pm 1, 0$ only acting on (e_1, e_2, e_3) and the lower 4×4 matrices involve the additional elements $\pm\frac{1}{2}$ and transform the octonionic units (e_4, e_5, e_6, e_7) among themselves. The upper block of 3×3 matrices in fact transforms the roots of $SU(2)\times SU(2)\times SU(2)$ where each $SU(2)$ is represented by one octonionic unit $\pm e_i$ ($i=1,2,3$) whereas the lower 4×4 matrices preserve the root system of an $SO(8)$ Lie algebra whose roots are the linear combination of e_is ($i=4,5,6,7$). Therefore, the group of order 96 with 16 conjugacy classes leaves the root system of $SU(2)^3\times SO(8)$ invariant where the roots are given by

$$\begin{aligned}
&SU(2){:}\quad \pm e_1\\
&SU(2){:}\quad \pm e_2\\
&SU(2){:}\quad \pm e_3\\
&SO(8){:}\quad \pm e_4,\ \pm e_5,\ \pm e_6,\ \pm e_7 \qquad \tfrac{1}{2}(\pm e_4\pm e_5\pm e_6\pm e_7)\,.
\end{aligned} \tag{13}$$

The matrices of $G_2'(2)$ which preserve this root system are generated by the matrices C and E. Again noting that M_4 preserves the same root system, the group is easily extended to a group of order 192 which is maximal in $G_2(2)$ and has 17 conjugacy classes. The $SU(2)^3\times SO(8)$ is not a maximal subalgebra as it is involved in $SU(2)\times SO(12)$. However, we should emphasize that while the $SU(2)$ in the case of $SU(2)\times SO(12)$ were invariant, here three $SU(2)$ roots are permuted. Once again we notice that the group 192′ involves three maximal subgroups of order 96 but with different structures. They have, respectively, 16, 13 and 19 conjugacy classes, and only the one with 16 classes is involved in $G_2'(2)$. The group of order 96 with 13 conjugacy classes has the structure $4^2.S_3$ where 4^2 denotes the direct product of two cyclic groups of order 4 and S_3 is the permutation group of 3-objects. It is understood that 4^2 is invariant in $4^2.S_3$. An extension of this to 192′ is made by the 2-group, so the structure is given by $4^2.D_6$ where D_6 is the dihedral group of order 12. In fact, D_6 is the automorphism group of the root system of G_2. The $4^2.D_6$ group has many interesting subgroups such as the binary tetrahedral group $2A_4$ of order 24, and dihedral groups of various orders. With a unitary transformation

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \to (e_1, e_2, e_3, u, d, d^*, -u^*) \tag{14}$$

where u, d are the split octonions

$$\begin{aligned} u&=\frac{1}{\sqrt{2}}(e_4+\mathrm{i}e_5) \qquad & u^*&=\frac{1}{\sqrt{2}}(e_4-\mathrm{i}e_5) \\ d&=\frac{1}{\sqrt{2}}(e_6+\mathrm{i}e_7) \qquad & d^*&=\frac{1}{\sqrt{2}}(e_6-\mathrm{i}e_7) \end{aligned} \tag{15}$$

the generators of 192′ can be written in the form

$$\left[\begin{array}{c|c} R_i & \\ \hline & Q_i \end{array}\right] \qquad i=1,2 \tag{16}$$

where

$$\begin{aligned} R_1&=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad & Q_1&=\frac{1}{\sqrt{2}}(\mathrm{i}\sigma_2+\mathrm{i}\sigma_3)\otimes\frac{1}{\sqrt{2}}(I+\mathrm{i}\sigma_1) \\ R_2&=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad & Q_2&=-\sqrt{2}(I+\mathrm{i}\sigma_1)\otimes\frac{1}{\sqrt{2}}(I+\mathrm{i}\sigma_3) \end{aligned} \tag{17}$$

and σ_i ($i=1,2,3$) are 2×2 Pauli matrices and $\otimes$ stands for the direct product of Q_1 and Q_2. If all possible products of R_1 and R_2 were allowed then we would have generated the direct products of two binary octahedral groups so that the order would be $48\times48=2304$ instead of 192.

We have checked that the 3×3 matrices R_1 and R_2 generate a group of order 24 isomorphic to S_4 with five conjugacy classes. This is the octahedral group where $\pm e_i$ ($i=1,2,3$) denote the vertices of the octahedron. However, the lower 4×4 blocks of matrices Q_1, Q_2 generate a group of order 192 which unifies the two two-dimensional irreducible representation of D_6 in the four-dimensional representation of the group 192′.

3.4. $PSL_2(7)$ and the root system of $SU(8)$

$SU(8)$ is a maximal subalgebra of E_7 where the E_7 roots decompose under $SU(8)$ as

$$126=56+70. \tag{18}$$

Here, 56 represents the non-zero roots of $SU(8)$ and 70 the weights of the 70-dimensional representation. We are interested in the subgroup of $G_2'(2)$ which preserves this decomposition. Octonionic roots of $SU(8)$ are given by

$$
\begin{aligned}
&\pm e_1, \pm e_2, \pm e_4, \pm e_6 \\
&\pm\tfrac{1}{2}(\pm e_1 \pm e_2 + e_5 + e_7) \\
&\pm\tfrac{1}{2}(\pm e_1 + e_3 \pm e_6 + e_7) \\
&\pm\tfrac{1}{2}(\pm e_1 + e_3 \pm e_4 + e_5) \\
&\pm\tfrac{1}{2}(\pm e_2 + e_3 - e_5 \pm e_6) \\
&\pm\tfrac{1}{2}(\pm e_2 + e_3 \pm e_4 - e_7) \\
&\pm\tfrac{1}{2}(\pm e_4 + e_5 \pm e_6 - e_7).
\end{aligned} \tag{19}
$$

The matrices A and E of $G_2'(2)$ preserve this system and generate a group of order 168 with six conjugacy classes (the well known finite subgroup of $SU(3)$). The 7×7 dimensional representation is irreducible and the characters of the conjugacy classes are given by table 3.

Table 3. Character values of the seven-dimensional irreducible representation of $PSL_2(7)$.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	24	0	7
3	24	0	7
4	21	−1	2
5	42	−1	4
6	56	1	3

The group 168 has the maximal subgroups S_4 and a group of order 21 with the structure 3:7. We will see that the group 21 will also play an important role in the construction of the group 1344 which admits another group of order 168 with eight conjugacy classes. We should emphasize that the group 168 which preserves the octonionic roots of $SU(8)$ is the finite subgroup of $SU(3)$ which is isomorphic to $PSL_2(7)$ [5, 15]. Its class structure and the characters of the seven-dimensional representation are the same as for $PSL_2(7)$. The matrix M_4 also preserves the root system of $SU(8)$. Therefore, $PSL_2(7)$ can be extended to the group $PSL_2(7){:}2$ of order 336 with nine conjugacy classes. Hence the generators of the group 336 are A, E and M_4. This group is related to the conformal field theory of G_2 affine Lie algebra [4, 5].

Another interesting observation here is that the group 336 is a subgroup of the Weyl group of $SU(8)$ with index 120.

Before we conclude this section we should remark on the $SU(3)\times SU(6)$ maximal subalgebra of E_7. The finite group which preserves the root system of $SU(3)\times SU(6)$ is a group of order 36 which turns out to be a subgroup of the group 436 as $SU(6)$ is contained in E_6. Therefore the automorphism group of the octonionic roots of $SU(3)\times SU(6)$ is not a maximal subgroup of $G_2(2)$.

4. The group 1344 as the automorphism group of pure octonionic units

This group is unconnected with the adjoint Chevalley group although it is one of the maximal subgroups of the finite Lie group $G_2(3)$. We have already stated that the root system of E_8 can be obtained in seven different ways, in each of which one of the octonionic unit e_i ($i=1,\ldots,7$) plays an important role. These seven sets of 240 integral octonions and 126 pure octonionic sets thereof can be obtained from (4) by cyclic permutations of e_i($i=1,\ldots,7$). As a consequence of this we observe that among the seven different octonionic presentations of the E_7 root system only the 14 octonionic units $\pm e_i$ ($i=1,\ldots,7$) which are common to all suggests to us that the group 192 which permutes only $\pm e_i$ can be extended to a larger group by adjoining the cyclic permutation of order 7. Indeed, the group turns out to be of order $192\times 7=1344$. However, this group is no longer the automorphism group of the octonionic roots of E_7. However, it permutes the octonionic roots of $SU(2)^7\subset E_7$, which can be represented by $\pm e_i$ ($i=1,\ldots,7$). Because the group 1344 permutes the roots of E_7 it is a subgroup of the Weyl group of E_7 with the index $9.240=2160$.

1344 can be generated, for example, by the matrices

$$F=\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad G=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

It has 11 conjugacy classes, and the characters of the seven-dimensional irreducible representation are given by table 4.

Table 4. Character values of the seven-dimensional irreducible representation of the group 1344.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	7	−1	2
3	42	−1	4
4	84	−1	2
5	42	3	4
6	168	−1	8
7	168	1	8
8	192	0	7
9	192	0	7
10	224	−1	6
11	224	1	3

1344 has three maximal subgroups, two of which have order 192 and the other order 168. One of the subgroups of order 192 is the usual one which preserves the roots of $SO(12)$ in E_7 and has 14 conjugacy classes. Indeed, it is the group which fixes $\pm e_1$. The second maximal subgroup of 1344 of order 192 has 13 classes.

The maximal subgroup of the group 1344 of order 168 has eight classes in contrast to $PSL_2(7)$, which only has six classes. The generators of the group 168 are given by the matrices

$$H=\begin{bmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0 \end{bmatrix} \qquad K=\begin{bmatrix} 0&-1&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&-1&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0 \end{bmatrix}. \tag{21}$$

The group 168 has the character values for the seven-dimensional irreducible representation given by table 5.

Table 5. Character values of the seven-dimensional irreducible representation of the group 168.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	7	−1	2
3	24	0	7
4	24	0	7
5	28	−1	6
6	28	1	3
7	28	−1	6
8	28	1	3

A comparison of tables 3 and 5 implies that the group 168 here is not isomorphic to the $PSL_2(7)$. The structure of the group 168 appears much simpler while acting on seven pure octonions. It can be generated by the following transformations:

(i) a cyclic permutation of $e_1e_2e_4e_3e_6e_5e_7$;

(ii) fix e_1 and let $(e_2 \to e_4 \to e_6)$ and $(e_3 \to e_7 \to e_5)$ be permuted;

(iii) the matrix M_4.

Transformations (i) and (ii) generate a group of order $21=3{:}7$. This group has five conjugacy classes, and $PSL_2(7)$ also has the same group as a maximal subgroup. The main distinction arises in the fact that although $PSL_2(7)$ admits S_4 as another maximal subgroup, the group 168 does not involve S_4 at all. Another feature is that the group 168 admits, as an invariant subgroup, the elementary Abelian group of order 8 generated by the matrices M_1, M_2 and M_4.

Transformation (i) and M_4 generate a maximal subgroup of the group 168 of order 56 with eight conjugacy classes. Here the elementary Abelian group 2^3 generated by M_1, M_2 and M_4 is the normal subgroup of the group 56. Therefore the structure of 168 is $2^3.7{:}3$, which is completely different from $PSL_2(7)$, a simple group which preserves the octonionic roots of $SU(8)$ in E_7. The group 1344 has many interesting smaller groups such as the binary tetrahedral group, S_4, dicyclic groups of orders 16 and 12, the dihedral group of order 8 and the quaternion group.

Another interesting group of order 1344 (which we denote by 1344′) arises as the semi-direct product of the Abelian group, of order 8, generatd by the matrices M_1, M_2 and M_4 by its automorphism group $PSL_2(7)$. To understand its structure, we will proceed as follows.

Let α permute $M_1, \ldots, M_7$ in the cyclic order ($\alpha^7 = I$) and β operate as $(M_1)(M_2M_4M_6)$ $(M_3M_7M_5)$. Then α and β generate the group $21 \approx 7{:}3$, which is isomorphic to one of the maximal subgroup of $PSL_2(7)$, and the maximal subgroup of the group 168 with eight conjugacy classes having the structure $2^3.7{:}3$.

Now let γ transform M_i ($i=1, \ldots, 7$) as $(M_1)(M_2M_4)(M_3M_7)(M_5)(M_6)$. Then α, β and γ ($\alpha^7 = \beta^3 = \gamma^2 = \beta^{-1}\alpha\beta\alpha^{-2} = (\beta\gamma)^2 = (\alpha\gamma)^2 = 1$) generate the group $PSL_2(7)$ of order 168 with six conjugacy classes. Indeed the seven-dimensional representation acting on M_i ($i=1, \ldots, 7$) is reducible because $\Sigma_{i=1}^{7} M_i = -1$ and it decomposes as $7 = 1 \oplus 6$ where 1 and 6 are the irreducible representation of $PSL_2(7)$. One can extend $PSL_2(7)$ to the group 1344′ by adjoining the matrices M_i.

The group 1344′ also has 11 conjugacy classes with the same character table as the group 1344, preserving the octonionic sets $\pm e_i$ ($i=1, \ldots, 7$). The character table of the seven-dimensional irreducible representation of the group 1344′ is given in table 6.

Table 6. Character values of the seven-dimensional irreducible representation of the group 1344′.

Class	No. matrices in a class	Trace	Power
1	1	7	1
2	7	−1	2
3	42	−1	2
4	84	−1	4
5	42	3	2
6	168	−1	4
7	168	1	4
8	192	0	7
9	192	0	7
10	224	−1	6
11	224	1	3

Although two groups are of the same order 1344 and have the same character values for the seven-dimensional irreducible representations, they are not isomorphic to each other. Comparing tables 4 and 6 one observes that the powers of some of the matrices do not match.

It is clear from the definition of α, β, γ that the Abelian group 2^3 generated by M_1, M_2 and M_4 is an invariant subgroup of the group 1344′. We have already noticed that α, β and M_i generated the group 168, with eight conjugacy classes, preserving the octonionic algebra of the set $\pm e_i$ ($i=1, \ldots, 7$). Therefore, the new group 1344′ possesses two maximal subgroups of order 168, one being isomorphic to $PSL_2(7)$ with six conjugacy classes the other isomorphic to the group 168 with eight conjugacy classes [11].

5. Conclusion

If octonions play any role in physics, the smallest exceptional group G_2, the automorphism group of the octonion algebra, should be the relevant symmetry of the physical system described by the octonions. Their relevance to E_8 and E_7 is already manifest. E_6 as a grand unified theory or E_8 as a gauge symmetry of superstring theories may require the group structures which we have discussed so far.

To the best of our knowledge the seven-dimensional irreducible representation of the adjoint Chevalley group $G_2(2)$ constructed by the inner automorphism of the integral octonions has not been obtained elsewhere. The correspondence between the octonionic subroot systems of E_7 and the maximal subgroups of $G_2(2)$, for the first time, is made clear in this work. We have observed that there is a close connection between the Weyl groups of these Lie algebras and $G_2(2)$ and their maximal subgroups:

$$\frac{|W(E_7)/Z_2|}{|G_2(2)|}=\frac{|W(E_6)|}{436}=\frac{|W(SU(8))|}{336}=\frac{|W(SO(12))|}{192}=120. \tag{22}$$

$G_2(2)$ and the groups of order 1344 arise as maximal subgroups in various groups of Lie type and sporadic groups. For instance, $G_2'(2)$ is one of the maximal subgroups of the Hall–Janko group J_2, and the group $2^3.\ PSL_2(7)$ of order 1344 plays an important role in the constructions of various sporadic groups. These features of the automorphism groups of octonions may relate sporadic groups to any physics associated with the exceptional Lie algebras E_8, E_7 and their subalgebras.

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Appendix A. Matrices generating $G_2(2)$ and its maximal subgroups

$$A=\begin{bmatrix}1&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&1&0&0\end{bmatrix}\qquad B=\frac{1}{2}\begin{bmatrix}0&-1&1&1&0&0&1\\1&0&1&0&0&1&-1\\-1&1&0&1&0&1&0\\1&0&-1&0&0&1&1\\0&0&0&0&2&0&0\\0&1&1&-1&0&0&1\\1&1&0&1&0&-1&0\end{bmatrix} \tag{23}$$

$$C=\begin{bmatrix}-1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&1\\0&0&0&-1&0&0&0\\0&0&0&0&0&-1&0\end{bmatrix}\qquad D=\begin{bmatrix}-1&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&-1\\0&0&-1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&-1&0&0&0\end{bmatrix} \tag{24}$$

$$E=\frac{1}{2}\begin{bmatrix}0&-2&0&0&0&0&0\\2&0&0&0&0&0&0\\0&0&2&0&0&0&0\\0&0&0&-1&1&1&-1\\0&0&0&-1&-1&-1&-1\\0&0&0&1&1&-1&-1\\0&0&0&1&-1&1&-1\end{bmatrix}\qquad M_4=\begin{bmatrix}-1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&-1\end{bmatrix}\tag{25}$$

$$L_4=\begin{bmatrix}-1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&-1&0&0&0\end{bmatrix}.\tag{26}$$

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